

A compactification of outer space which is an absolute retract

Mladen Bestvina and Camille Horbez*

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Abstract

We define a new compactification of outer space CV_N (the *Pacman compactification*) which is an absolute retract, for which the boundary is a Z -set. The classical compactification \overline{CV}_N made of very small F_N -actions on \mathbb{R} -trees, however, fails to be locally 4-connected as soon as $N \geq 4$. The Pacman compactification is a blow-up of \overline{CV}_N , obtained by assigning an orientation to every arc with nontrivial stabilizer in the trees.

Introduction

The study of the group $\text{Out}(F_N)$ of outer automorphisms of a finitely generated free group has greatly benefited from the study of its action on Culler–Vogtmann’s outer space CV_N . It is therefore reasonable to look for compactifications of CV_N that have “nice” topological properties. The goal of the present paper is to construct a compactification of CV_N which is a compact, contractible, finite-dimensional absolute neighborhood retract (ANR), for which the boundary is a Z -set.

One motivation for finding “nice” actions of a group G on absolute retracts comes from the problem of solving the Farrell–Jones conjecture for G , see [2, Section 1]. For instance, it was proved in [4] that the union of the Rips complex of a hyperbolic group together with the Gromov boundary is a compact, contractible ANR, and this turned out to be a crucial ingredient in the proof by Bartels–Lück–Reich of the Farrell–Jones conjecture for hyperbolic groups [2]. A similar approach was recently used by Bartels to extend these results to the context of relatively hyperbolic groups [1].

We review some terminology. A compact metrizable space X is said to be an *absolute (neighborhood) retract* (AR or ANR) if for every compact metrizable space Y that contains X as a closed subset, the space X is a (neighborhood) retract of Y . Given $x \in X$, we say that X is *locally contractible (LC) at x* if for every open neighborhood \mathcal{U}

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of x , there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that the inclusion map $\mathcal{V} \hookrightarrow \mathcal{U}$ is nullhomotopic. More generally, X is *locally n -connected (LC^n) at x* if for every open neighborhood \mathcal{U} of x , there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that for every $0 \leq i \leq n$, every continuous map $f : S^i \rightarrow \mathcal{V}$ from the i -sphere is nullhomotopic in \mathcal{U} . We then say that X is *LC* (or *LC^n*) if it is *LC* (or *LC^n*) at every point $x \in X$.

A nowhere dense closed subset Z of a compact metrizable space X is a *Z-set* if X can be instantaneously homotoped off of Z , i.e. if there exists a homotopy $H : X \times [0, 1] \rightarrow X$ so that $H(x, 0) = x$ and $H(X \times (0, 1]) \subseteq X \setminus Z$. Given $z \in Z$, we say that Z is *locally complementarily contractible (LCC) at z* , resp. *locally complementarily n -connected (LCC^n) at z* , if for every open neighborhood \mathcal{U} of z in X , there exists a smaller open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of z in X such that the inclusion $\mathcal{V} \setminus Z \hookrightarrow \mathcal{U} \setminus Z$ is nullhomotopic in X , resp. trivial in π_i for all $0 \leq i \leq n$. We then say that Z is *LCC* (resp. *LCC^n*) in X if it is *LCC* (resp. *LCC^n*) at every point $z \in Z$.

Every ANR space is locally contractible. Further, if X is an ANR, and $Z \subseteq X$ is a Z-set, then Z is *LCC* in X . Conversely, it is a classical fact that every finite-dimensional, compact, metrizable, locally contractible space X is an ANR, and further, an n -dimensional *LC^n* compact metrizable space is an ANR, see [20, Theorem V.7.1]. If X is further assumed to be contractible then X is an AR. In Appendix B of the present paper, we will establish (by similar methods) a slight generalization of this fact, showing that if X is an n -dimensional compact metrizable space, and $Z \subseteq X$ is a nowhere dense closed subset which is *LCC^n* in X , and such that $X \setminus Z$ is *LC^n* , then X is an ANR and Z is a Z-set in X .

Culler–Vogtmann’s *outer space* CV_N can be defined as the space of all F_N -equivariant homothety classes of free, minimal, simplicial, isometric F_N -actions on simplicial metric trees (with no valence 2 vertices). Culler–Morgan’s compactification of outer space [7] can be described by taking the closure in the space of all F_N -equivariant homothety classes of minimal, nontrivial F_N -actions on \mathbb{R} -trees, equipped with the equivariant Gromov–Hausdorff topology. The closure $\overline{CV_N}$ identifies with the space of homothety classes of minimal, *very small* F_N -trees [6, 3, 19], i.e. those trees whose arc stabilizers are cyclic and root-closed (possibly trivial), and whose tripod stabilizers are trivial.

When $N = 2$, the closure $\overline{CV_2}$ was completely described by Culler–Vogtmann in [9]. The closure of reduced outer space (where one does not allow for separating edges in the quotient graphs) is represented on the left side of Figure 1: points in the circle at infinity represent actions dual to measured foliations on a once-punctured torus, and there are “spikes” coming out corresponding to simplicial actions where some edges have nontrivial stabilizer. These spikes prevent the boundary $\overline{CV_2} \setminus CV_2$ from being a Z-set in $\overline{CV_2}$: these are locally separating subspaces in $\overline{CV_2}$, and therefore $\overline{CV_2}$ is not *LCC* (it is not even *LCC^0*) at points on these spikes. More surprisingly, while $\overline{CV_2}$ is an absolute retract (and we believe that so is $\overline{CV_3}$), this property fails as soon as $N \geq 4$.

Theorem 1. *For all $N \geq 4$, the space $\overline{CV_N}$ is not locally 4-connected, hence it is not an AR.*

There are however many trees in $\overline{CV_N}$ at which $\overline{CV_N}$ is locally contractible: for example, we prove in Section 1.2 that $\overline{CV_N}$ is locally contractible at any tree with trivial arc stabilizers. The reason why local 4-connectedness fails in general is the following. When $N \geq 4$, there are trees in $\overline{CV_N}$ that contain both a subtree dual to an arational measured foliation on a nonorientable surface Σ of genus 3 with a single boundary curve c , and a simplicial edge with nontrivial stabilizer c . We construct such a tree T_0 (see

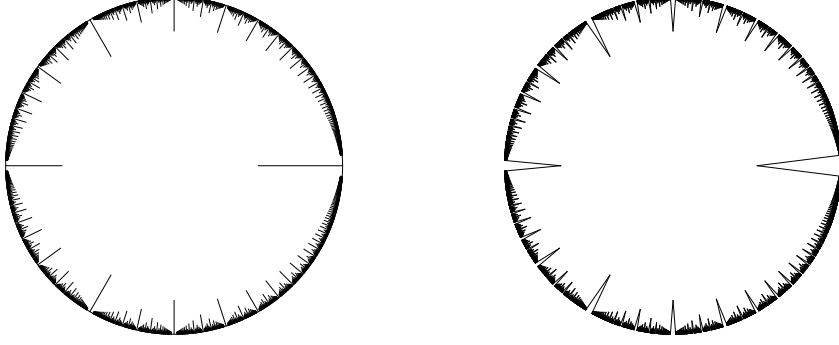


Figure 1: The reduced parts of the classical compactification \overline{CV}_2 (on the left), and of the Pacman compactification \widehat{CV}_2 (on the right).

Section 1 for its precise definition), at which \overline{CV}_N fails to be locally 4-connected, due to the combination of the following two phenomena.

- The space $X(c)$ of trees where c fixes a nondegenerate arc locally separates \overline{CV}_N at T_0 .
- The subspace of $\mathcal{PMF}(\Sigma)$ made of foliations which are dual to very small F_N -trees contains arbitrarily small embedded 3-spheres which are not nullhomologous: these arise as $\mathcal{PMF}(\Sigma')$ for some orientable subsurface $\Sigma' \subseteq \Sigma$ which is the complement of a Möbius band in Σ . Notice here that a tree dual to a curve on Σ may fail to be very small, in the case where the curve represents a conjugacy class in $\pi_1(\Sigma)$ which is a square (arising as the boundary of a Möbius band in Σ).

We will find an open neighborhood \mathcal{U} of T_0 in \overline{CV}_N such that for any smaller neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 , we can find a 3-sphere S^3 in $X(c) \cap \mathcal{V}$ (provided by the second point above) which is not nullhomologous in $X(c) \cap \mathcal{U}$, but which can be capped off by balls B_{\pm}^4 in each of the two complementary components of $X(c) \cap \mathcal{V}$ in \mathcal{V} . By gluing these two balls along their common boundary S^3 , we obtain a 4-sphere in \mathcal{V} , which is shown not to be nullhomotopic within \mathcal{U} by appealing to a Čech homology argument, presented in Appendix A of the paper.

It would be of interest to have a better understanding of the topology of the space $\mathcal{PMF}(\Sigma)$ in order to have a precise understanding of the failure of local connectivity of \overline{CV}_N .

Question: Is \overline{CV}_N locally 3-connected for every N ? Is \overline{CV}_3 locally contractible?

However, we also build a new compactification \widehat{CV}_N of CV_N (a blow-up of \overline{CV}_N) which is an absolute retract, for which the boundary is a Z -set. The remedy to the bad phenomena described above is to prescribe orientations in an F_N -equivariant way to all arcs with nontrivial stabilizers in trees in \overline{CV}_N , which has the effect in particular to “open up” \overline{CV}_N at the problematic spaces $X(c)$. In other words, the characteristic set of any element $g \in F_N \setminus \{e\}$ in a tree T is given an orientation as soon as it is not reduced to a point: when g acts as a hyperbolic isometry of T , its axis comes with a natural orientation, and we also decide to orient the edges with nontrivial stabilizers. Precise definitions of \widehat{CV}_N and its topology are given in Section 2 of the present paper. In rank

2, this operation has the effect of “cutting” along the spikes (see Figure 1), which leads us to call this new compactification the *Pacman compactification* of outer space.

Theorem 2. *The space $\widehat{CV_N}$ is an absolute retract of dimension $3N-4$, and $\widehat{CV_N} \setminus CV_N$ is a Z -set.*

The space $\widehat{CV_N}$ is again compact, metrizable and finite-dimensional: this is established in Section 2 of the present paper from the analogous results for $\overline{CV_N}$. Also, we show in Section 3.5 that every point in $\widehat{CV_N}$ is a limit of points in CV_N . The crucial point for proving Theorem 2 is to show that the boundary $\widehat{CV_N} \setminus CV_N$ is locally complementarily contractible.

The proof of this last fact is by induction on the rank N , and the strategy is the following. Given a tree $T \in \widehat{CV_N} \setminus CV_N$, one can first approximate T by trees that split as graphs of actions over free splittings of F_N , and admit 1-Lipschitz F_N -equivariant maps to T . Using our induction hypothesis (and working in the outer space of each of the factors that are elliptic in the splitting), we prove that subspaces in $\widehat{CV_N}$ made of trees that split as graphs of actions over a given free splitting are locally complementarily contractible at every point in the boundary. We also find a continuous way of deforming a neighborhood of T into one of these subspaces, so that T is sent to a nearby tree. This enables us to prove that $\widehat{CV_N} \setminus CV_N$ is locally complementarily contractible at T .

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1 The space $\overline{CV_N}$ is not an AR when $N \geq 4$.

1.1 Review: Outer space and Culler–Morgan’s compactification

Outer space and its closure. Let $N \geq 2$. *Outer space* CV_N (resp. *unprojectivized outer space* cv_N) is the space of F_N -equivariant homothety (resp. isometry) classes of simplicial, free, minimal, isometric F_N -actions on simplicial metric trees, with no valence 2 vertices. Unprojectivized outer space can be embedded into the space of all F_N -equivariant isometry classes of minimal F_N -actions on \mathbb{R} -trees, which is equipped with the *equivariant Gromov–Hausdorff topology* introduced in [26, 27]. This is the topology for which a basis of open neighborhoods of a tree T is given by the sets $\mathcal{N}_T(K, X, \varepsilon)$ (where $K \subseteq T$ is a finite set of points, $X \subseteq F_N$ is a finite subset, and $\varepsilon > 0$), defined in the following way: an F_N -tree T' belongs to $\mathcal{N}_T(K, X, \varepsilon)$ if there exists a finite set $K' \subseteq T'$ and a bijection $K \rightarrow K'$ such that for all $x, y \in K$ and all $g \in X$, one has $|d_{T'}(x', gy') - d_T(x, gy)| < \varepsilon$ (where x', y' are the images of x, y under the bijection). The closure $\overline{cv_N}$ was identified in [6, 3, 19] with the space of F_N -equivariant isometry classes of minimal, very small actions of F_N on \mathbb{R} -trees (an action is called *very small* if arc

stabilizers are cyclic and root-closed [possibly trivial], and tripod stabilizers are trivial). Note that we allow for the trivial action of F_N on a point in $\overline{cv_N}$. The compactification $\overline{CV_N}$ is the space of homothety classes of nontrivial actions in $\overline{cv_N}$. In the present paper, for carrying induction arguments, we will need to allow for the case where $N = 1$, in which case cv_1 is the collection of all possible isometry classes of \mathbb{Z} -actions on the real line (these are just parameterized by the translation length of the generator), and $\overline{cv_1}$ is obtained by adding the trivial action on a point.

Morphisms between F_N -trees and a semi-flow on $\overline{CV_N}$. A *morphism* between two F_N -trees T and T' is an F_N -equivariant map $f : T \rightarrow T'$, such that every segment $I \subseteq T$ can be subdivided into finitely many subsegments I_1, \dots, I_k , so that for all $i \in \{1, \dots, k\}$, the map f is an isometry when restricted to I_i . A morphism $f : T \rightarrow T'$ is *optimal* if in addition, for every $x \in T$, there is an open arc $I \subseteq T$ around x on which f is one-to-one. We denote by \mathcal{A} the space of isometry classes of all F_N -trees, and by $\text{Opt}(\mathcal{A})$ the space of optimal morphisms between trees in \mathcal{A} , which is equipped with the equivariant Gromov–Hausdorff topology, see [16, Section 3.2]. The following statement can be found in [16, Section 3], it is based on work of Skora [29] inspired by an idea of Steiner.

Proposition 1.1. (*Skora [29], Guirardel–Levitt [16]*) *There exist continuous maps $H : \text{Opt}(\mathcal{A}) \times [0, 1] \rightarrow \mathcal{A}$ and $\Phi, \Psi : \text{Opt}(\mathcal{A}) \times [0, 1] \rightarrow \text{Opt}(\mathcal{A})$ such that*

- *for all $f \in \text{Opt}(\mathcal{A})$, we have $\Phi(f, 0) = \text{id}$ and $\Psi(f, 0) = f$,*
- *for all $f \in \text{Opt}(\mathcal{A})$, we have $\Phi(f, 1) = f$ and $\Psi(f, 1) = \text{id}$,*
- *for all $f \in \text{Opt}(\mathcal{A})$ and all $t \in [0, 1]$, we have $\Psi(f, t) \circ \Phi(f, t) = f$, and*
- *for all $f \in \text{Opt}(\mathcal{A})$ and all $t \in [0, 1]$, the tree $H(f, t)$ is the range of the morphism $\Phi(f, t)$.*

The proof of Proposition 1.1 goes as follows: given a morphism $f : T_0 \rightarrow T_1$, one first defines for all $t \in [0, 1]$ a minimal F_N -tree T_t , as the quotient space T_0/\sim_t , where $a \sim_t b$ whenever $f(a) = f(b)$ and $\tau(a, b) := \sup_{c \in [a, b]} d_{T_1}(f(a), f(c)) \leq t$. The morphism f factors through optimal morphisms $\phi_t : T_0 \rightarrow T_t$ and $\psi_t : T_t \rightarrow T_1$ which vary continuously with f . The path $(H(f, t))_{t \in [0, 1]}$ will be called the *canonical folding path* directed by f .

Lemma 1.2. *Assume that f is isometric when restricted to any arc of T_0 with nontrivial stabilizer. Then for all $t \in [0, 1]$, the map $\psi_t : T_t \rightarrow T_1$ is isometric when restricted to arcs of T_t with nontrivial stabilizer.*

Proof. Let $[a_t, b_t] \subseteq T_t$ be a nondegenerate arc with nontrivial stabilizer c . We aim to show that $\psi_t(a_t) \neq \psi_t(b_t)$, which is enough to conclude since ψ_t is a morphism. By definition of T_t , there exist $a, b \in T_0$ satisfying $\tau(a, ca) \leq t$ and $\tau(b, cb) \leq t$, with $f(ca) = f(a)$ and $f(cb) = f(b)$, such that $a_t = \phi_t(a)$ and $b_t = \phi_t(b)$.

Assume towards a contradiction that $\psi_t(a_t) = \psi_t(b_t)$. Then $f(a) = f(b)$. If c does not fix any nondegenerate arc in T_0 , then the segment $[a, b]$ is contained in the union of all c^k -translates of $[a, ca]$ and $[b, cb]$, with k varying over \mathbb{Z} . It follows that $\tau(a, b) \leq \max\{\tau(a, ca), \tau(b, cb)\} \leq t$, and hence $a_t = b_t$, a contradiction. Assume now that c fixes a nondegenerate arc $[a', b'] \subseteq T_0$, and let a'' (resp. b'') be the projection of a (resp. b) to $[a', b']$. Using the fact that f is isometric when restricted to

$[a', b']$, we have $f([a'', b'']) \subseteq f([a, a'']) \cup f([b, b''])$, and therefore we get that $\tau(a, b) = \max\{\sup_{x \in [a, a'']} d_{T_1}(f(a), f(x)), \sup_{y \in [b, b'']} d_{T_1}(f(b), f(y))\}$. Since $[a, a''] \subseteq [a, ca]$ and $[b, b''] \subseteq [b, cb]$, we then obtain as above that $\tau(a, b) \leq t$, so again $a_t = b_t$, a contradiction. \square

Remark 1.3. Together with [17, Proposition 4.4], which says that arc stabilizers in the intermediate trees are root-closed if arc stabilizers are root-closed in T_0 and T_1 , Lemma 1.2 implies that if $T_0, T_1 \in \overline{cv_N}$, and if f is isometric when restricted to arcs with nontrivial stabilizer, then all intermediate trees belong to $\overline{cv_N}$. It is also a classical fact that if $T_0, T_1 \in cv_N$, then all intermediate trees belong to cv_N .

1.2 Local contractibility at points with trivial arc stabilizers

The following lemma provides nice approximations of trees in $\overline{cv_N}$ with trivial arc stabilizers, it is proved by the same argument as Proposition 3.3 below.

Lemma 1.4. ([18, Theorem 6.3]) *Given a tree $T \in \overline{cv_N}$ with all arc stabilizers trivial, and any open neighborhood \mathcal{U} of T in $\overline{cv_N}$, there exists a tree $U \in \mathcal{U} \cap cv_N$ that admits an optimal morphism onto T .*

Proof. The arguments as in Proposition 3.3 produce a simplicial tree $U' \in \mathcal{U}$ and an optimal morphism $U' \rightarrow T$. However, U' may not be free (but it has trivial arc stabilizers because T has trivial arc stabilizers). One way to replace U' by a tree in cv_N is to replace the V -fixed point v by a free V -tree wedged at the point v , and perform this operation equivariantly and at all nontrivial fixed vertices. The natural map $U' \rightarrow U$ will collapse these trees and will not be a morphism. However, attaching the trees at nearby points fixes this problem. \square

Lemma 1.5. *Let $f : S \rightarrow T$ be an optimal morphism from a tree $S \in cv_N$ to a tree $T \in \overline{cv_N}$. Then there is a neighborhood \mathcal{W} of T and a continuous map*

$$\Psi_S : \mathcal{W} \rightarrow \text{Opt}(\overline{cv_N})$$

such that for all $W \in \mathcal{W}$, the source of $\Psi_S(W)$ is a tree $S' \in cv_N$, in the same (cone on a) simplex as S and varying continuously, the range of $\Psi_S(W)$ is the tree W , and $\Psi_S(W)$ is an optimal morphism. In addition, $\Psi_S(T) = f$.

Remark 1.6. Lemma 1.5 yields in particular a continuous choice of a basepoint in every tree in \mathcal{W} , obtained as the image of a fixed vertex in S .

Proof. Let v be a vertex of S . Since f is optimal, the point v belongs to a line $l \subseteq S$ such that the restriction $f|_l$ is an isometry (notice however that we may not assume in general that l is the axis of an element of F_N). We can then find two hyperbolic elements $\gamma_{1,v}, \gamma_{2,v} \in F_N$ whose axes in T both intersect $f(l)$ but do not intersect each other, and such that the segment joining $Ax_T(\gamma_{1,v})$ to $Ax_T(\gamma_{2,v})$ contains $f(v)$ in its interior. Let $d \in \mathbb{R}$ denote the distance from $Ax_T(\gamma_{1,v})$ to $f(v)$.

If W is sufficiently close to T , the elements $\gamma_{1,v}$ and $\gamma_{2,v}$ are hyperbolic in W and their axes are disjoint and lie at distance at least d from each other. We denote by x_W^v the point at distance d from $Ax_W(\gamma_{1,v})$ on the segment from $Ax_W(\gamma_{1,v})$ to $Ax_W(\gamma_{2,v})$. Given a choice v_1, \dots, v_k of representatives of the orbits of the vertices of S , there is a unique choice of a (new) metric on S , giving a tree S_W , so that the linear extension of g_W (defined on vertices by sending v_i to $x_W^{v_i}$, and extending equivariantly) is a morphism

(some edges may get length 0, but then restrict to a smaller neighborhood). Using the fact that $f|_l$ is an isometry, we get that this morphism is also optimal: indeed, letting v_1 and v_2 be the two vertices of S adjacent to v on the line l , then the segment joining $Ax_T(\gamma_{1,v})$ to $Ax_T(\gamma_{2,v})$ contains $[f(v_1), f(v_2)]$. It follows that when W is close to T the segment joining $Ax_W(\gamma_{1,v})$ to $Ax_W(\gamma_{2,v})$ overlaps $[g_W(v_1), g_W(v_2)]$ in a segment that contains $g_W(v)$ in its interior. In particular, g_W sends the two directions at v determined by l to distinct directions. Thus we set $\Psi_S(W) = g_W$. It is standard that Ψ_S is continuous, see [15] for example. \square

Lemma 1.7. *Let $T \in \overline{cv_N}$, let \mathcal{U} be an open neighborhood of T in $\overline{cv_N}$, and let $\varepsilon > 0$. Then there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T in $\overline{cv_N}$ such that if $U \in \mathcal{V}$ is a tree that admits a $(1 + \varepsilon)$ -Lipschitz F_N -equivariant map onto T , and if $U' \in \overline{cv_N}$ is a tree such that f factors through $(1 + \varepsilon)$ -Lipschitz F_N -equivariant maps from U to U' and from U' to T , then $U' \in \mathcal{U}$.*

Proof. We can find an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T in $\overline{cv_N}$ defined as the set of trees T' satisfying conditions of the form

$$(1 - \delta)\|g_i\|_T < \|g_i\|_{T'} < (1 + \delta)\|g_i\|_T$$

for some $\delta > 0$ and some finite set $\{g_1, \dots, g_k\} \subseteq F_N$. If $U \in \mathcal{V}$ is a tree as in the statement of the lemma, then for all $i \in \{1, \dots, k\}$, we have $\|g_i\|_T < (1 + \varepsilon)\|g_i\|_{U'} < (1 + \varepsilon)^2\|g_i\|_U < (1 + \varepsilon)^2(1 + \delta)\|g_i\|_T$. By choosing \mathcal{V} sufficiently small, we can therefore arrange that $U' \in \mathcal{U}$. \square

Proposition 1.8. *The space $\overline{cv_N}$ is locally contractible at every tree with all arc stabilizers trivial.*

Proof. Let $T \in \overline{cv_N}$ be a tree with all arc stabilizers trivial, and let \mathcal{U} be an open neighborhood of T in $\overline{cv_N}$. Let $\varepsilon > 0$, and let $\mathcal{V} \subseteq \mathcal{U}$ be a smaller neighborhood of T provided by Lemma 1.7. Let $S \in \mathcal{V} \cap cv_N$ be such that there exists an optimal morphism from S to T (this exists by Lemma 1.4). Then there exists a smaller neighborhood $\mathcal{W} \subseteq \mathcal{V}$ of T such that for all $T' \in \mathcal{W}$, there is a $(1 + \varepsilon)$ -Lipschitz F_N -equivariant map from S to the source S' of the morphism $\Psi_S(T')$ given by Lemma 1.5. In view of Lemma 1.7, this implies that all trees that belong to either the straight path from S to S' , or to the canonical folding path directed by $\Psi_S(T')$, belong to \mathcal{U} . As $\Psi_S(T')$ varies continuously with T' , this gives a homotopy of \mathcal{W} onto S that stays within \mathcal{U} . \square

A variant of the above argument shows the following statement, which will be useful in our proof of the fact that $\overline{CV_N}$ is not an AR.

Lemma 1.9. *Let $T_0 \in \overline{cv_N}$ be a tree with trivial arc stabilizers that is dual to a measured foliation on a surface Σ with a single boundary component c . Let $Z \subseteq \overline{cv_N}$ be the set of all trees dual to a measured foliation on Σ .*

Then for every open neighborhood \mathcal{V} of T_0 in $\overline{cv_N}$, there exist a smaller neighborhood $\mathcal{W} \subseteq \mathcal{V}$ of T_0 in $\overline{cv_N}$, a tree $S \in \mathcal{W} \cap cv_N$, and a homotopy $H : (Z \cap \mathcal{W}) \times [0, 1] \rightarrow \mathcal{V}$ such that $H(z, 0) = z$ and $H(z, 1) = S$ for all $z \in Z \cap \mathcal{W}$, and c is hyperbolic in $H(z, t)$ for all $z \in Z$ and all $t > 0$.

Proof. The proof of Lemma 1.9 is the same as the proof of Proposition 1.8, except that we have to show in addition that c remains hyperbolic until it reaches z along the canonical folding path from S' to z determined by the morphism $\Psi_S(z)$. Notice

that c is not contained in any proper free factor of F_N , so whenever c becomes elliptic along a canonical folding path, the tree T reached by the path contains no edge with trivial stabilizer. In view of Lemma 1.2, no two simplicial edges with nontrivial stabilizer can get identified by the folding process. In addition, an arc in a subtree with dense orbits from the Levitt decomposition of T as a graph of actions [24] cannot get identified with a simplicial edge with nontrivial stabilizer, and two such arcs cannot either get identified together [18, Lemmas 1.9 and 1.10]. This implies that the canonical folding path $(H(\Psi_S(z), t))_{t \in [0,1]}$ becomes constant once it reaches T . To conclude the proof of Lemma 1.9, it remains to reparametrize this canonical folding path to ensure that it does not reach T before $t = 1$. Notice that $\|c\|_{H(\Psi_S(z), t)}$ decreases strictly as t increases, until it becomes equal to 0, and in addition the tree $H(\Psi_S(z), t)$ is the same for all $t \in [0, 1]$ such that $\|c\|_{H(\Psi_S(z), t)} = 0$. We can therefore reparametrize the canonical folding path by the translation length of c : for all $l \leq \|c\|_{H(\Psi_S(z), 0)}$, we let $H'(\Psi_S(z), \|c\|_{H(\Psi_S(z), 0)} - l)$ be the unique tree T on the folding path for which $\|c\|_T = l$. To get a homotopy from $(Z \cap \mathcal{W}) \times [0, 1]$ to \mathcal{V} , we then renormalize the parameter l by dividing it by $\|c\|_{H(\Psi_S(z), 0)}$. \square

1.3 The space $\overline{CV_N}$ is not locally 4-connected when $N \geq 4$.

We will now prove that Culler–Morgan’s compactification $\overline{CV_N}$ of outer space is not an AR as soon as $N \geq 4$.

Theorem 1.10. *For all $N \geq 4$, the space $\overline{CV_N}$ is not locally 4-connected, hence it is not an AR.*

Remark 1.11. It can actually be shown however that the closure $\overline{CV_2}$ is an absolute retract, and we also believe that $\overline{CV_3}$ is an absolute retract, though establishing this fact certainly requires a bit more work than the arguments from the present paper.

An embedded 3-sphere in $\mathcal{PMF}^{vs}(\Sigma)$. Throughout the present section, we let Σ be a nonorientable surface of genus 3 with one boundary component (so that its Euler characteristic is -2). We denote by $\mathcal{PMF}^{vs}(\Sigma)$ the subspace of $\mathcal{PMF}(\Sigma)$ made of all measured foliations that are dual to very small trees. Let γ be a simple closed curve on Σ that separates Σ into a Möbius band and an orientable surface Σ_0 (which is a compact surface of genus 1 with two boundary components). Then the space $\mathcal{PMF}(\Sigma_0)$ is a subset of $\mathcal{PMF}^{vs}(\Sigma)$, however $\gamma \notin \mathcal{PMF}^{vs}(\Sigma)$ because γ is a square in $\pi_1(\Sigma)$. The key observation for constructing 4-spheres in $\overline{CV_N}$ showing that $\overline{CV_N}$ is not locally 4-connected will be the following.

Lemma 1.12. *The subset $\mathcal{PMF}(\Sigma_0) \subseteq \mathcal{PMF}^{vs}(\Sigma)$ is a 3-sphere which is a retract of $\mathcal{PMF}^{vs}(\Sigma)$.*

Proof. The space $\mathcal{PMF}(\Sigma_0)$ is a topologically embedded 3-dimensional sphere in the 4-dimensional sphere $\mathcal{PMF}(\Sigma)$, so it separates $\mathcal{PMF}(\Sigma)$. Both sides of $\mathcal{PMF}(\Sigma) \setminus \mathcal{PMF}(\Sigma_0)$ contain curves arising as the boundary of a Möbius band, for which the dual tree is not very small (notice that one of the two sides is a cone of $\mathcal{PMF}(\Sigma_0)$ over γ). This implies that $\mathcal{PMF}(\Sigma_0)$ is a retract of $\mathcal{PMF}^{vs}(\Sigma)$. \square

Definition of the tree T_0 . We will now define a tree $T_0 \in \overline{CV_N}$ at which $\overline{CV_N}$ will fail to be locally 4-connected. Recall that a *splitting* of F_N is a simplicial F_N -tree, and that a

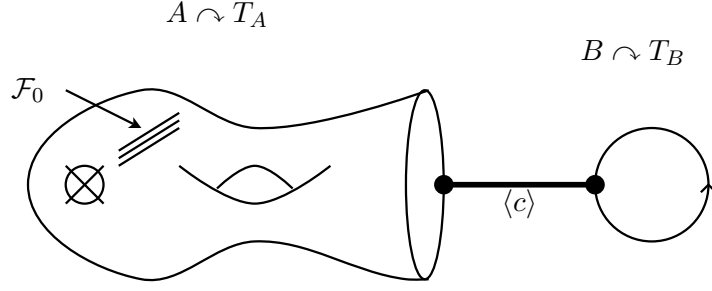


Figure 2: The tree T_0 at which \overline{CV}_N fails to be locally 4-connected.

tree $U \in \overline{cv}_N$ is said to *split as a graph of actions* over a splitting S if U is obtained from S by equivariantly attaching actions $G_v \curvearrowright T_v$ to every vertex $v \in S$ with stabilizer G_v , with prescribed attaching points for the adjacent edges, see [24] for a precise definition.

Let $N \geq 4$. Let $T_0 \in \overline{CV}_N$ be the tree defined in the following way (see Figure 2). Let \mathcal{F}_0 be an arational measured foliation on Σ , obtained as the attracting foliation of a pseudo-Anosov diffeomorphism f of Σ . Let A be the fundamental group of Σ , which is free of rank 3, let T_A be the very small A -tree dual to \mathcal{F}_0 , and let $x_A \in T_A$ be the unique point fixed by a nontrivial element c_A corresponding to the boundary curve of Σ .

Let $B = F_{N-2}$, which we write as a free product $B = B' * \langle c_B \rangle$ for some element $c_B \in B$. Let $T_B \in \overline{cv}_{N-2}$ be a tree that splits as a graph of actions over this free splitting of B , with vertex actions a free and simplicial action $T_{B'} \in \overline{cv}_{N-3}$, and the trivial action of c_B on a point, where the edge with trivial stabilizer from the splitting is given length 0. Let x_B be the point fixed by c_B in T_B , and notice that c_B belongs to the B' -minimal subtree $T_{B'}$ of T_B .

Write $F_N = A *_{c_A=c_B} B$, and let T_0 be the very small F_N -tree obtained as a graph of actions over this amalgamated free product, with vertex actions T_A and T_B and attaching points x_A and x_B , where the simplicial edges with nontrivial stabilizers coming from the splitting are assigned length 1. We let $c := c_A = c_B \in F_N$.

Finding embedded 4-spheres in a neighborhood of T_0 . Recall here that the *characteristic set* $\text{Char}_T(g)$ of an element $g \in F_N$ in an F_N -tree T is its axis if g is hyperbolic and its fixed point set if g is elliptic. When T is very small the characteristic set of a nontrivial elliptic element is a closed interval (possibly a point). Moreover, if characteristic sets of g and h intersect in more than a point in T , then the same is true in a neighborhood of T in \overline{CV}_N .

Choose an element $g \in F_N$ of the form $g = ab'$ with $a \in A$ and $b' \in B'$, which is hyperbolic in T_0 and whose axis crosses the arc fixed by c . Let \mathcal{U} be an open neighborhood of T_0 in \overline{CV}_N consisting of trees where g is hyperbolic, the characteristic sets of g and c overlap, and the surface group A is not elliptic. The crucial property satisfied by such a neighborhood \mathcal{U} of T_0 is that we have the *oriented translation length* of c , namely the continuous function

$$\theta : \mathcal{U} \rightarrow \mathbb{R}$$

defined by

$$\theta(T) = \varepsilon_T \frac{\|c\|_T}{\|g\|_T}$$

where $\varepsilon_T = 1$ (resp. -1) if c is hyperbolic in T and the axes of c and g give the same

(resp. opposite) orientation to the overlap, and $\varepsilon_T = 0$ if c is elliptic. We will denote by \mathcal{U}_+ (resp. \mathcal{U}_-) the subset of \mathcal{U} made of trees such that $\theta(T) \geq 0$ (resp. $\theta(T) \leq 0$). Similarly, given any smaller neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 , we will let $\mathcal{V}_+ := \mathcal{V} \cap \mathcal{U}_+$ and $\mathcal{V}_- := \mathcal{V} \cap \mathcal{U}_-$.

Let \mathcal{U}_A be an open neighborhood of T_A in $\overline{cv(A)}$ such that there exists an element $a \in A$ such that the characteristic sets of c and a have empty intersection in all trees in \mathcal{U}_A . For all $U_A \in \mathcal{U}_A$, we then denote by $b(U_A)$ the projection of $\text{Char}_{U_A}(a)$ to $\text{Char}_{U_A}(c_A)$: this gives us a continuous choice of a basepoint in every tree $U_A \in \mathcal{U}_A$. Given a tree $U_A \in \overline{cv(A)}$ in which c_A is hyperbolic, and $t \in \mathbb{R}$, we let $b(U_A, t)$ be the unique point on the axis of c_A at distance $|t|$ from $b(U_A)$, and such that $[b(U_A), b(U_A, t)]$ is oriented in the same direction as the axis of c_A if $t \geq 0$, and in the opposite direction if $t \leq 0$. We then let $\mathcal{T}(U_A, t) \in \overline{cv(A)}$ be the tree which splits as a graph of actions over the free product $F_N = A * B'$, with vertex actions U_A and $T_{B'}$, and attaching points $b(U_A, t)$ and x_B , where the simplicial edge from the splitting is assigned length 0. The following lemma follows from the argument in the proof of [19, Lemma 5.6], we also refer to Section 3.6 of the present paper, for a similar argument.

Lemma 1.13. *For every open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 in $\overline{CV_N}$, there exists an open neighborhood $\mathcal{V}_A \subseteq \mathcal{U}_A$ of T_A in $\overline{cv(A)}$ such that for all $U_A \in \mathcal{V}_A$ such that c_A is hyperbolic in U_A , we have $\mathcal{T}(U_A, 1) \in \mathcal{V}_+$, and $\mathcal{T}(U_A, -1) \in \mathcal{V}_-$. \square*

Denote by $X(c)$ the subspace of $\overline{CV_N}$ made of all trees where c is elliptic, and by $X(c)^*$ the subset of $X(c)$ consisting of trees where the surface group A is not elliptic. Notice that our choice of neighborhood \mathcal{U} of T_0 ensures that $\mathcal{U} \cap X(c) \subseteq X(c)^*$. For any tree T in $X(c)^*$, the tree T_A is dual to some measured foliation on Σ , so we have a map $\psi : X(c)^* \rightarrow \mathcal{PMF}^{vs}(\Sigma)$.

Proposition 1.14. *For every open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 , there exists a topologically embedded 3-sphere S^3 in $X(c) \cap \mathcal{V}$ which is a retract of $X(c) \cap \mathcal{U}$, and such that the inclusion map $\iota : S^3 \hookrightarrow X(c) \cap \mathcal{V}$ extends to continuous maps $\iota_{\pm} : B_{\pm}^4 \rightarrow \mathcal{V}_{\pm}$ (where B_{\pm}^4 are 4-balls with boundary S^3), with $\iota_{\pm}(B_{\pm}^4 \setminus S^3) \subseteq \overline{CV_N} \setminus X(c)$.*

Proof. We identify $\mathcal{PMF}(\Sigma)$ with a continuous lift in $\mathcal{MF}(\Sigma)$. This gives a continuous injective map $\phi : \mathcal{PMF}^{vs}(\Sigma) \rightarrow X(c)^*$, mapping every measured foliation \mathcal{F} to the tree in $\overline{CV_N}$ that splits as a graph of actions over $F_N = A *_c B$, with vertex actions the tree dual to \mathcal{F} , and the fixed action $B \curvearrowright T_B$, where the simplicial edge from the splitting is assigned length 1. The composition $\psi\phi$ is the identity. Thus given an orientable subsurface $\Sigma_0 \subset \Sigma$ as in Lemma 1.12, the 3-sphere $\phi(\mathcal{PMF}(\Sigma_0))$ is a retract of $X(c) \cap \mathcal{U}$ (notice also that its image in $\overline{cv(A)}$ is again an embedded 3-sphere which we denote by S_A^3). To ensure that this 3-sphere is contained in \mathcal{V} , use uniform north-south dynamics of the pseudo-Anosov homeomorphism f on $\mathcal{PMF}(\Sigma)$ to find $k \in \mathbb{Z}$ so that $\phi(\mathcal{PMF}(f^k(\Sigma_0))) \subseteq X(c) \cap \mathcal{V}$ (see [21, Theorem 3.5], which is stated there in the case of an orientable surface, but the similar statement for a nonorientable surface follows by considering the orientable double cover of Σ).

Let now \mathcal{V}_A be an open neighborhood of T_A in $\overline{cv(A)}$ provided by Lemma 1.13, and let $\mathcal{W}_A \subseteq \mathcal{V}_A$ be a smaller neighborhood of T_A provided by Lemma 1.9. We can assume that the sphere constructed in the above paragraph is such that $S_A^3 \subseteq \mathcal{W}_A$. By Lemma 1.9, there exists a tree $V_A \in \mathcal{V}_A \cap \overline{cv(A)}$ and a homotopy $H_A : S_A^3 \times [0, 1] \rightarrow \mathcal{V}_A$ such that

- for all $z \in S_A^3$, we have $H_A(z, 0) = z$, and $H_A(z, 1) = V_A$, and

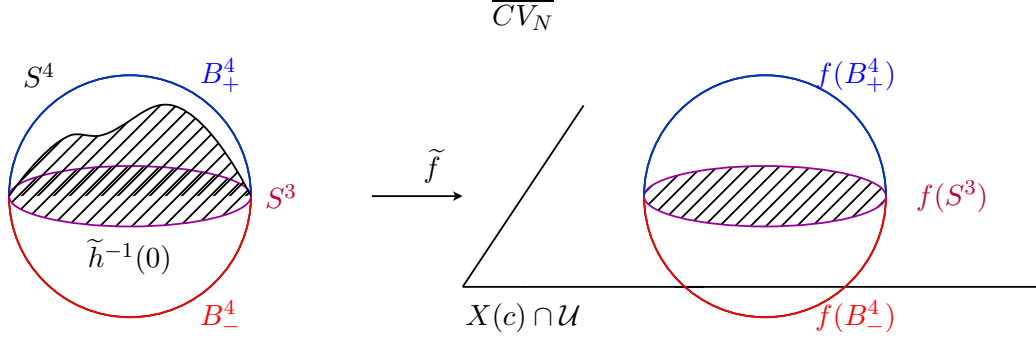


Figure 3: Constructing the sphere S^4 showing failure of local 4-connectivity of \overline{CV}_N .

- for all $z \in S_A^3$ and all $t > 0$, the element c is hyperbolic in $H_A(z, t)$.

We now define $H_{\pm} : S^3 \times [0, 1] \rightarrow \mathcal{V}_{\pm}$ by letting $H_{\pm}(z, 0) = z$ for all $z \in S^3$, and $H_{\pm}(z, t) = \mathcal{T}(H_A(z, t), \pm 1)$ for all $z \in S^3$ and all $t > 0$. Continuity follows from the argument from [19, Lemma 4.6] (see also Section 3.4 below, in particular Lemma 3.8). This enables us to construct the maps ι_{\pm} , where the segment joining a point $z \in S^3$ to the center of B_{\pm}^4 , is mapped to $H_{\pm}(z, [0, 1])$. \square

Our proof of Theorem 1.10 uses a Čech homology argument, given in Appendix A of the paper.

Proof of Theorem 1.10. We will show that for every neighborhood $\mathcal{V} \subset \mathcal{U}$ of T_0 there is a map $S^4 \rightarrow \mathcal{V}$ which is not nullhomotopic in \mathcal{U} . This is illustrated in Figure 3. Let S^3 be a topologically embedded 3-sphere in $X(c) \cap \mathcal{V}$ provided by Proposition 1.14. In particular, the inclusion map $\iota : S^3 \hookrightarrow X(c) \cap \mathcal{U}$ is nontrivial in homology (either singular homology, or Čech homology \check{H}_3 with $\mathbb{Z}/2$ coefficients). In addition, we can extend ι to maps $\iota_{\pm} : B_{\pm}^4 \rightarrow \mathcal{V}_{\pm}$, with $\iota_{\pm}(B_{\pm}^4 \setminus S^3) \subseteq \overline{CV}_N \setminus X(c)$. Gluing these two along the boundary produces a map $f : S^4 \rightarrow \mathcal{V}$, which restricts to ι on S^3 , and such that the composition $h = \theta f : S^4 \rightarrow \mathbb{R}$ (where we recall that θ is the oriented translation length of c) is standard (i.e. $h^{-1}(0) = S^3$, $h^{-1}([0, +\infty)) = B_+^4$ and $h^{-1}((-\infty, 0]) = B_-^4$).

Now suppose that f extends to $\tilde{f} : B^5 \rightarrow \mathcal{U}$, and let $\tilde{h} = \theta \tilde{f} : B^5 \rightarrow \mathbb{R}$. If $\tilde{h}^{-1}(0)$ were a manifold with boundary S^3 , then we would immediately deduce that the inclusion

$$S^3 \hookrightarrow \tilde{h}^{-1}(0)$$

is trivial in singular homology H_3 . This may fail to be true, but we can still apply Lemma A.1 to deduce that the above inclusion is trivial in Čech homology \check{H}_3 with $\mathbb{Z}/2$ coefficients. Applying \tilde{f} and observing that $\tilde{f}(\tilde{h}^{-1}(0)) \subseteq X(c) \cap \mathcal{U}$ we deduce that

$$f|_{S^3} = \iota : S^3 \rightarrow X(c) \cap \mathcal{U}$$

is trivial in \check{H}_3 , a contradiction. \square

2 The Pacman compactification of outer space

In the present paper, we will be interested in another compactification \widehat{CV}_N of outer space, which we call the *Pacman compactification*. We also define \widehat{cv}_N as the unprojectivized version of \widehat{CV}_N . We will now define \widehat{CV}_N and establish some basic topological properties.

A point in $\widehat{CV_N}$ is given by a tree $T \in \overline{CV_N}$, together with an F_N -equivariant choice of orientation of the characteristic set of every nontrivial element of F_N that fixes a nondegenerate arc in T (notice that the axis of every hyperbolic element also comes with a natural orientation). Precisely, given any nontrivial element $c \in F_N$ which is not a proper power, and whose characteristic set in T is a nondegenerate arc $\text{Char}_T(c) := [x, y]$ fixed by c , we prescribe an orientation of $\text{Char}_T(c)$, and the orientation of $\text{Char}_T(c^{-1})$ is required to be opposite to the orientation of $\text{Char}_T(c)$. The F_N -translates of $[x, y]$ get the induced orientation as required by F_N -equivariance. We will write the oriented arc as $x \xrightarrow{c} y$. Notice that there is an F_N -equivariant surjective map $\pi : \widehat{CV_N} \rightarrow \overline{CV_N}$, which consists in forgetting the orientations of the edges with nontrivial stabilizer.

Given two oriented (possibly finite or infinite) geodesics l and l' in an \mathbb{R} -tree T with nondegenerate intersection, we define the *relative orientation* of l and l' as being equal to $+1$ if the orientations of l and l' agree on their intersection, and -1 otherwise. Given $T \in \widehat{CV_N}$, and two elements $\alpha, \beta \in F_N$ whose characteristic sets have nondegenerate intersection in T , we define the *relative orientation* of the pair (α, β) in T as being equal to the relative orientation of their characteristic sets.

We now define a topology on $\widehat{CV_N}$. Given an open set $U \subseteq \overline{CV_N}$, and a finite collection of pairs $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$ of elements of F_N , such that for all $i \in \{1, \dots, k\}$, β_i is hyperbolic and the characteristic sets of α_i and β_i have nondegenerate intersection in all trees in U , we let

$$U((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$$

be the set of all $T \in \widehat{CV_N}$ such that $\pi(T) \in U$, and the relative orientation of (α_i, β_i) in T is equal to $+1$ for all $i \in \{1, \dots, k\}$. Notice that given two open sets $U, U' \subseteq \overline{CV_N}$, and finite collections of elements $\alpha_i, \beta_i, \alpha'_j, \beta'_j \in F_N$, the intersection $U((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)) \cap U'((\alpha'_1, \beta'_1), \dots, (\alpha'_k, \beta'_k))$ is equal to $(U \cap U')((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k), (\alpha'_1, \beta'_1), \dots, (\alpha'_k, \beta'_k))$. This shows the following lemma.

Lemma 2.1. *The sets $U((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$ form a basis of open sets for a topology on $\widehat{CV_N}$.* \square

From now on, we will equip $\widehat{CV_N}$ with the topology generated by these sets. For this topology, the map $\pi : \widehat{CV_N} \rightarrow \overline{CV_N}$ is continuous.

Proposition 2.2. *The space $\widehat{CV_N}$ is second countable.*

Proof. Since $\overline{CV_N}$ is second countable, we can choose a countable basis $(U_i)_{i \in \mathbb{N}}$ of open neighborhoods of $\overline{CV_N}$. Then the countable collection made of all sets of the form $U_i((\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$ (where for all $j \in \{1, \dots, k\}$, the element β_j is hyperbolic and the characteristic sets of α_j and β_j have nondegenerate intersection in all trees in U_i) is a basis of open neighborhoods of $\widehat{CV_N}$. \square

Proposition 2.3. *The space $\widehat{CV_N}$ is Hausdorff.*

Proof. Let $T \neq T' \in \widehat{CV_N}$. If $\pi(T) \neq \pi(T')$, then since $\overline{CV_N}$ is Hausdorff, we can find disjoint open neighborhoods U of $\pi(T)$ and U' of $\pi(T')$ in $\overline{CV_N}$, and these yield disjoint open neighborhoods of T and T' in $\widehat{CV_N}$. So we can assume that $\pi(T) = \pi(T')$, and there is an arc e with nontrivial stabilizer c in $\pi(T)$ whose orientation is not the same in T and T' . Let $g, g' \in F_N$ be two elements whose characteristic sets in $\pi(T)$ lie in each of the two complementary components of e . Then gg' is hyperbolic in $\pi(T)$ and its axis crosses e . One can thus find an open neighborhood U of $\pi(T)$ in $\overline{CV_N}$ such

that the characteristic sets of gg' and c have nondegenerate intersection in all trees in U . Then exactly one of the points $T, T' \in \widehat{CV_N}$ belongs to $U(c, gg')$, the other belongs to $U(c, g'g)$, and $U(c, gg') \cap U(c, g'g) = \emptyset$. This shows that $\widehat{CV_N}$ is Hausdorff. \square

Proposition 2.4. *The inclusion map $CV_N \hookrightarrow \widehat{CV_N}$ is a topological embedding.*

Proof. This follows from the analogous result for the inclusion $CV_N \hookrightarrow \overline{CV_N}$. Indeed, if $(S_n)_{n \in \mathbb{N}} \in CV_N^{\mathbb{N}}$ escapes every compact subset of CV_N , then using compactness of $\overline{CV_N}$ (established in [7, Theorem 4.5]), we can assume up to passing to a subsequence that $(S_n)_{n \in \mathbb{N}}$ converges to a tree $T \in \overline{CV_N}$, and $T \notin CV_N$. Then all accumulation points of $(S_n)_{n \in \mathbb{N}}$ in $\widehat{CV_N}$ must project to T under the continuous map π , in particular $(S_n)_{n \in \mathbb{N}}$ does not converge to any point in CV_N . \square

The following lemma gives a useful criterion for checking that a sequence converges in $\widehat{CV_N}$.

Lemma 2.5. *Let $(T_n)_{n \in \mathbb{N}} \in \widehat{CV_N}^{\mathbb{N}}$, and let $T \in \widehat{CV_N}$. Assume that $(\pi(T_n))_{n \in \mathbb{N}}$ converges to $\pi(T)$ in $\overline{CV_N}$, and that for all elements $\alpha \in F_N$ fixing a nondegenerate arc in T , there exists a hyperbolic element β whose axis in T contains $\text{Char}_T(\alpha)$, such that the relative orientations of (α, β) eventually agree in T_n and in T . Then $(T_n)_{n \in \mathbb{N}}$ converges to T .*

Remark 2.6. One can equivalently replace the assumption that the axis of β contains $\text{Char}_T(\alpha)$, by the assumption that the axis of β has nondegenerate intersection with $\text{Char}_T(\alpha)$.

Proof. Since $(\pi(T_n))_{n \in \mathbb{N}}$ converges to $\pi(T)$, it is enough to prove that given any two elements $\alpha, \beta' \in F_N$ whose characteristic sets in T have nondegenerate intersection, the relative orientations of (α, β') eventually agree in T_n and in T . This is clearly true if either α and β' are both hyperbolic in T , or if they are both elliptic (in which case their characteristic sets in T have degenerate intersection, unless they are both powers of a common element or of an element and its inverse). So assume that α fixes a nondegenerate arc in T , and that β' is hyperbolic in T . By hypothesis, there exists $\beta \in F_N$ for which the condition holds. The axes of β and β' will eventually have nondegenerate intersection in T_n , and their relative orientations will eventually agree in T_n and in T . This implies that the relative orientations of (α, β') eventually agree in T_n and in T . \square

Proposition 2.7. *The space $\widehat{CV_N}$ is compact.*

Proof. Since $\widehat{CV_N}$ is second countable, it is enough to prove sequential compactness. Let $(T_n)_{n \in \mathbb{N}} \in \widehat{CV_N}^{\mathbb{N}}$. Since $\overline{CV_N}$ is compact [7, Theorem 4.5], up to passing to a subsequence, we can assume that $(\pi(T_n))_{n \in \mathbb{N}}$ converges to a tree $T \in \overline{CV_N}$.

Let $\alpha \in F_N$ be an element that fixes a nondegenerate arc in T , and let $\beta \in F_N$ be a hyperbolic element in T , whose axis has nondegenerate intersection with $\text{Char}_T(\alpha)$. Then up to passing to a subsequence, we can assume that the relative orientation of (α, β) in T_n is eventually constant, and assign the corresponding orientation to $\text{Char}_T(\alpha)$. If we do this equivariantly for each of the finitely many orbits of maximal arcs with nontrivial stabilizer in T , Lemma 2.5 ensures that we have found a subsequence of $(T_n)_{n \in \mathbb{N}}$ that converges to T . \square

Being compact, Hausdorff, and second countable, the space $\widehat{CV_N}$ is metrizable.

Corollary 2.8. *The space $\widehat{CV_N}$ is metrizable.* \square

Proposition 2.9. *The space $\widehat{CV_N}$ has finite topological dimension equal to $3N - 4$.*

Proof. The map $\pi : \widehat{CV_N} \rightarrow \overline{CV_N}$ is a continuous map between compact metrizable spaces, with finite point preimages. It follows from Hurewicz's theorem (see [12, Theorem 4.3.6]) that $\dim(\widehat{CV_N}) \leq \dim(\overline{CV_N})$. In addition, the space $\widehat{CV_N}$ contains CV_N as a topologically embedded subspace (Proposition 2.4). As $\dim(\overline{CV_N}) = \dim(CV_N) = 3N - 4$ (see [3, 13]), the result follows. \square

Proposition 3.9 below will show in addition that CV_N is dense in $\widehat{CV_N}$, so $\widehat{CV_N}$ is a compactification of CV_N . The $\text{Out}(F_N)$ -action clearly extends to this compactification.

3 The space $\widehat{CV_N}$ is an AR, and the boundary is a Z -set.

The goal of the present section is to prove Theorem 2 from the introduction. Sections 3.1 to 3.3 are adaptations to our context of classical arguments that were recalled in Section 1 for the classical closure $\overline{cv_N}$. The key point of the proof of Theorem 2 is to prove that the boundary $\widehat{cv_N} \setminus cv_N$ is locally complementarily contractible in cv_N at any tree T_0 (we refer to the introduction for a definition). This is done by first deforming an open neighborhood \mathcal{U} of T_0 into a subspace of $\widehat{cv_N}$ made of trees that split as graphs of actions over a one-edge free splitting of F_N (Section 3.4), and then arguing by induction on the rank, by working in each of the vertex actions of the splitting (Section 3.6). Using similar arguments, we also prove in Section 3.5 that cv_N is dense in $\widehat{cv_N}$.

3.1 Morphisms between trees in $\widehat{cv_N}$ and extension of the semi-flow

A *morphism* between two trees $T, T' \in \widehat{cv_N}$ is a morphism between $\pi(T)$ and $\pi(T')$ which is further assumed to be isometric and orientation-preserving when restricted to every edge with nontrivial stabilizer. It is *optimal* if the corresponding morphism from $\pi(T)$ to $\pi(T')$ is optimal. The space $\text{Mor}(\widehat{cv_N})$ of all morphisms between trees in $\widehat{cv_N}$ is topologized by saying that two morphisms are close whenever the corresponding morphisms between the projections of the trees in $\overline{cv_N}$ are close, and in addition the sources and ranges of the morphisms are close in $\widehat{cv_N}$. We note that the canonical map $\text{Mor}(\widehat{cv_N}) \rightarrow \text{Mor}(\overline{cv_N})$ is bounded-to-one, and it is injective when restricted to morphisms between two fixed trees. We denote by $\text{Opt}(\widehat{cv_N})$ the space of all optimal morphisms between trees in $\widehat{cv_N}$. We now extend Proposition 1.1 to morphisms between trees in $\widehat{cv_N}$.

Proposition 3.1. *There exist continuous maps $\widehat{H} : \text{Opt}(\widehat{cv_N}) \times [0, 1] \rightarrow \widehat{cv_N}$ and $\widehat{\Phi}, \widehat{\Psi} : \text{Opt}(\widehat{cv_N}) \times [0, 1] \rightarrow \text{Opt}(\widehat{cv_N})$ such that*

- for all $f \in \text{Opt}(\widehat{cv_N})$, we have $\widehat{\Phi}(f, 0) = \text{id}$ and $\widehat{\Psi}(f, 0) = f$,
- for all $f \in \text{Opt}(\widehat{cv_N})$, we have $\widehat{\Phi}(f, 1) = f$ and $\widehat{\Psi}(f, 1) = \text{id}$,
- for all $f \in \text{Opt}(\widehat{cv_N})$ and all $t \in [0, 1]$, we have $\widehat{\Psi}(f, t) \circ \widehat{\Phi}(f, t) = f$, and
- for all $f \in \text{Opt}(\widehat{cv_N})$ and all $t \in [0, 1]$, the tree $\widehat{H}(f, t)$ is the range of the morphism $\widehat{\Phi}(f, t)$.

Proof of Proposition 3.1. Let $f \in \text{Opt}(\widehat{cv_N})$ be a morphism with source T_0 and range T_1 . Proposition 1.1 and Lemma 1.2 enable us to define $\widehat{H}(f, t)$ for all $(f, t) \in \text{Opt}(\widehat{cv_N}) \times [0, 1]$, by pulling back the orientations on the edges of T_1 in all trees $H(f, t)$. We also get morphisms $\widehat{\Phi}(f, t)$ and $\widehat{\Psi}(f, t)$. We will check that the map \widehat{H} defined in this way is continuous, from which it follows that $\widehat{\Phi}$ and $\widehat{\Psi}$ are also continuous.

Let $(f, t) \in \text{Opt}(\widehat{cv_N}) \times [0, 1]$, and let $((f_n, t_n))_{n \in \mathbb{N}} \in (\text{Opt}(\widehat{cv_N}) \times [0, 1])^{\mathbb{N}}$ be a sequence that converges to (f, t) . By Proposition 1.1, the sequence $(\pi(\widehat{H}(f_n, t_n)))_{n \in \mathbb{N}}$ converges to $\pi(\widehat{H}(f, t))$. Let $\alpha \in F_N$ be an element that fixes a nondegenerate arc e in $\widehat{H}(f, t)$. Then α also fixes a nondegenerate arc in the range T of f . Since the morphism $\Psi(f, t)$ is optimal, there exists $\beta \in F_N$ which is hyperbolic in $\widehat{H}(f, t)$ and whose axis crosses the image of e , and such that no fold in $\widehat{H}(f, t)$ involves both the edge e and a subsegment of the axis of β . This implies that the relative orientations of (α, β) in $\widehat{H}(f, t)$ and in T are the same. For all $n \in \mathbb{N}$, let T_n be the range of the morphism f_n . Since $(T_n)_{n \in \mathbb{N}}$ converges to T , the relative orientations of (α, β) in T_n and in T eventually agree. Since the characteristic sets of α and β have nondegenerate overlap in all trees $\widehat{H}(f, t')$ with $t' \geq t$, a compactness argument shows that for $n \in \mathbb{N}$ large enough, the characteristic sets of α and β have nondegenerate overlap in all trees $\widehat{H}(f_n, t')$ with $t' \geq t_n$. Therefore, the relative orientation of (α, β) cannot change along the path from $\widehat{H}(f_n, t_n)$ to T_n , so it is the same in $\widehat{H}(f_n, t_n)$ and in T_n . This implies that the relative orientations of (α, β) eventually agree in $\widehat{H}(f_n, t_n)$ and in $\widehat{H}(f, t)$. Lemma 2.5 then shows that $\widehat{H}(f_n, t_n)$ converges to $\widehat{H}(f, t)$. \square

3.2 Approximations by trees that split over free splittings

The following lemma is a version of Lemma 1.7 for $\widehat{cv_N}$, which easily follows from the version in $\overline{cv_N}$.

Lemma 3.2. *Let $T \in \widehat{cv_N}$, and let \mathcal{U} be an open neighborhood of T in $\widehat{cv_N}$. Then there exists an open neighborhood $\mathcal{W} \subseteq \mathcal{U}$ of T in $\widehat{cv_N}$ such that if $U \in \mathcal{W}$ is a tree that admits a morphism f onto T , and if $U' \in \widehat{cv_N}$ is a tree such that f factors through morphisms from U to U' and from U' to T , then $U' \in \mathcal{U}$.* \square

A *one-edge free splitting* of F_N is the Bass–Serre tree of a graph of groups decomposition of F_N , either as a free product $F_N = A * B$, or as an HNN extension $F_N = A *$. If $T \in \widehat{cv_N}$ is a tree that splits as a graph of actions over a splitting S of F_N , we say that an attaching point of a vertex action T_v is *admissible* if either it belongs to the G_v -minimal subtree of T_v , or else it is an endpoint of an arc with nontrivial stabilizer contained in G_v . The following proposition extends the analogous result for $\overline{cv_N}$ (see [18, Theorem 3.11]).

Proposition 3.3. *Let $T \in \widehat{cv_N}$, and let \mathcal{U} be an open neighborhood of T . Then there exists a tree $U \in \mathcal{U}$ that splits as a graph of actions over a one-edge free splitting S of F_N with admissible attaching points, coming with an optimal morphism $f : U \rightarrow T$ such that every arc with nontrivial stabilizer in T is the f -image of an arc with nontrivial stabilizer in U , and such that f is an isometry when restricted to the minimal subtree of any vertex action.*

Proof. The proof of Proposition 3.3 builds on classical approximation techniques of very small F_N -trees, we present a sketch of the argument.

If $T \in \widehat{cv_N}$ is nongeometric (i.e. not dual to a measured foliation on a 2-complex [25]), then one can approximate T by a geometric tree $U \in \mathcal{U}$ that contains a simplicial

edge e with trivial stabilizer and admits an optimal morphism $f : U \rightarrow T$, keeping edge stabilizers the same (so that orientations can be assigned in a coherent way in the approximation). The edge e is dual to a one-edge free splitting S of F_N , and U splits as a graph of actions over S . Up to slightly folding e (and slightly increasing its length), we can assume that attaching points are admissible. By folding within every vertex action U_A , the morphism f factors through a tree U' in such a way that the induced morphism from U' to T is an isometry when restricted to the minimal subtree U_A^{min} . In view of Lemma 3.2, we can also ensure that $U' \in \mathcal{U}$.

We now assume that $T \in \widehat{cv_N}$ is geometric. Every geometric F_N -tree splits as a graph of actions with indecomposable vertex actions, which are either dual to arational measured foliations on surfaces, or of Levitt type.

If T contains a Levitt component, we approximate it by free and simplicial actions by first running the Rips machine, and then cutting along a little arc transverse to the foliation in a naked band, see [3, 14]: this remains true in $\widehat{cv_N}$ because each of these indecomposable trees has trivial arc stabilizers. This gives an approximation of T by a tree U that splits as a graph of actions over a one-edge free splitting of F_N , and comes equipped with an optimal morphism onto T . We then get the required conditions on f as in the nongeometric case.

Finally, one is left with the case where T is geometric, and all its indecomposable subtrees are dual to measured foliations on surfaces. In this situation, either some indecomposable tree has an *unused* boundary component in the sense that it satisfies the conclusion of [3, Lemma 4.1], or else T splits as a graph of actions over a one-edge free splitting of F_N [19, Proposition 5.10]. In the first situation, we can again cut along a little arc with extremity on the unused boundary component, and transverse to the foliation, to approximate the indecomposable component. The required condition on f is obtained as above. \square

3.3 Continuous choices of basepoints

Given $T_0 \in \widehat{cv_N}$, the following lemma enables us to choose basepoints continuously in all trees in a neighborhood of T_0 , with a prescribed choice on T_0 . In the statement, we fix a Cayley tree R of F_N with respect to a free basis of F_N , and a vertex $* \in R$, and we denote by $\text{Map}(F_N, \widehat{cv_N})$ the collection of all F_N -equivariant maps from a tree obtained from R by possibly varying edge lengths, to trees in $\widehat{cv_N}$. The following lemma is a version of Lemma 1.5 (the choice of a basepoint is independent from a choice of orientations), we rephrase it here for future reference.

Lemma 3.4. *Let $N \geq 2$, let $T_0 \in \widehat{cv_N}$, let $c \in F_N$, and let $x_0 \in T_0$ be a point that belongs to the characteristic set of c . There exists a neighborhood \mathcal{U} of T_0 and a continuous map $b : \mathcal{U} \rightarrow \text{Map}(F_N, \widehat{cv_N})$ such that for all $T \in \mathcal{U}$, the range of $b(T)$ is T , and the point $b(T)(*)$ belongs to the characteristic set of c in T , and in addition $b(T_0)(*) = x_0$.*

In the sequel, we will often abuse notation and write $b(T)$ instead of $b(T)(*)$ to refer to the point in the tree T .

Remark 3.5. When $N = 1$ and $T \in \widehat{cv_1}$, we just let $b(T)(*)$ be any point in T .

3.4 Definition of $X_{S,U}(T_0)$ and construction of the deformation

From now on, we fix a tree $T_0 \in \widehat{cv_N}$, and an open neighborhood \mathcal{U} of T_0 in $\widehat{cv_N}$. Let $\mathcal{W} \subseteq \mathcal{U}$ be a smaller neighborhood of T_0 provided by Lemma 3.2. We choose a one-edge free splitting S of F_N , and a tree $U \in \mathcal{W}$ that splits as a graph of actions over S

with admissible attaching points, provided by Proposition 3.3. In particular, there is an optimal morphism $f : U \rightarrow T$ such that $f|_{U_A^{\min}}$ is an isometry for every vertex group A of S with vertex action U_A (here U_A^{\min} denotes the A -minimal subtree of U_A). Every attaching point $u \in U$ of a vertex action $A \curvearrowright U_A$ either belongs to U_A^{\min} (in this case we let $c_u = e$), or else it is an endpoint of an arc with nontrivial stabilizer c_u , with the convention that the characteristic set of c_u in U is oriented pointing outwards U_A^{\min} . We let $x_u := f(u)$, and we let b_u be a map on $\widehat{cv(A)}$ provided by Lemma 3.4, associated to c_u and to the projection of x_u to $U_A^{\min} \subseteq T_0$ (this map is defined in a neighborhood \mathcal{U}_A of U_A^{\min} , which we can assume to be the same for all attaching points of U_A). We make the following definition.

Definition 3.6. *The set $X_{S,U}(T_0) \subseteq \widehat{cv_N}$ is the set of all trees T that split as a graph of actions over S , with the condition that for every vertex action $A \curvearrowright T_A$ and every attaching point $u \in U_A$,*

- *if $c_u \neq e$, then the corresponding attaching point in T_A belongs to the characteristic set of c_u , and*
- *if $c_u = e$, then the corresponding attaching point in T_A is $b_u(T_A^{\min})$.*

The goal of the present section is to define a continuous deformation from a neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 onto a subset of $X_{S,U}(T_0) \cap \mathcal{U}$.

Proposition 3.7. *Let $T_0 \in \widehat{cv_N}$, let \mathcal{U} be an open neighborhood of T_0 in $\widehat{cv_N}$, and let S, U be chosen as above. There exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 in $\widehat{cv_N}$, and a continuous map*

$$H : \mathcal{V} \times [0, 1] \rightarrow \mathcal{U}$$

such that for all $T \in \mathcal{V}$, one has $H(T, 0) = T$ and $H(T, 1) \in X_{S,U}(T_0)$, and in addition $H((cv_N \cap \mathcal{V}) \times [0, 1]) \subseteq cv_N$.

We will then let $\rho(T) := H(T, 1)$ for all $T \in \mathcal{V}$. We will present the proof of Proposition 3.7 in the case where S is the Bass–Serre tree of a splitting of F_N as a free product $F_N = A * B$, and then explain how to adapt the construction in the case of HNN extensions.

Case 1: *The free splitting S is the Bass–Serre tree of a decomposition of F_N as a free product $F_N = A * B$.*

We will define $\rho(T)$ as a tree that splits as a graph of actions over S , in which the minimal subtrees of the vertex actions are the A - and B -minimal subtrees of T , and our main concern is to make appropriate choices of basepoints.

Let $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{U}$ be an open neighborhood of T_0 such that for all $T \in \mathcal{V}$, we have $T_A^{\min} \in \mathcal{U}_A$ and $T_B^{\min} \in \mathcal{U}_B$ (where we recall the definition of the neighborhoods \mathcal{U}_A and \mathcal{U}_B from the paragraph preceding Definition 3.6). We let T_A be the tree obtained from T_A^{\min} by possibly adding the characteristic sets in T of all the conjugates of c_A (this can only add up to one orbit of extra edges to T_A^{\min}), and we define T_B similarly. If $c_A = e$, we let $b'_A(T) := b_A(T)$ (where we denote by b_A the choice of basepoint b_u associated to the attaching point for U_A). If $c_A \neq e$, then we let $b'_A(T)$ be either the projection of $\text{Char}_T(c_B)$ to $\text{Char}_T(c_A)$ if these sets are disjoint, or else the midpoint of their intersection. Notice that $b'_A(T) \in T_A$. We make similar definitions on the B side.

The tree $\rho(T)$ is defined as the tree that splits as a graph of actions over S with vertex actions T_A and T_B and attaching points $b'_A(T)$ and $b'_B(T)$, where the simplicial edge from the splitting is assigned length $d_T(b'_A(T), b'_B(T))$. There is a morphism $f_T : \rho(T) \rightarrow T$, which is the identity when restricted to T_A or T_B , and sends the simplicial edge linearly onto its image in T . Notice that the basepoints $b'_A(T)$ and $b'_B(T)$ depend continuously on T . Notice also that our construction implies that $\rho(T_0) = U$ and $f_{T_0} = f$: indeed, the attaching point in $\rho(T_0)$ and in U is the same, equal to x_u if $c_A = e$, and to the extremity of the arc fixed by c_A otherwise (and the same is also true on the B side).

Lemma 3.8. *The morphism $f_T : \rho(T) \rightarrow T$ depends continuously on the tree $T \in \mathcal{V}$.*

Proof. We need to establish that the tree $\rho(T)$ depends continuously on T . Once this is established, continuity of $T \mapsto f_T$ follows from the construction. Continuity of $\pi(\rho(T)) \in \overline{cv_N}$ follows from the construction and Guirardel's Reduction Lemma from [14, Section 4], see also the argument from the proof of [19, Lemma 5.6].

Notice that all nontrivial arc stabilizers in $\rho(T)$ are conjugate into either A or B . By Lemma 2.5, we only need to associate to any element $\alpha \in F_N$ fixing a nondegenerate arc in $\rho(T)$, an element $\beta \in F_N$ that is hyperbolic in $\rho(T)$ and whose axis has nondegenerate intersection with $\text{Char}_T(\alpha)$, such that the relative orientations of (α, β) eventually agree in $\rho(T_n)$ and in $\rho(T)$. This is clear if α fixes a nondegenerate arc in either the A - or the B -minimal subtree of T . If these minimal subtrees have empty intersection in $\rho(T)$, and α fixes a nondegenerate arc – say $\alpha = c_A$ – on the segment that joins them in T , then one can choose for β an element which is a product ab with $a \in A$ and $b \in B$. Indeed, the relative orientation of $[b_A(T'), b'_A(T')]$ and of $\text{Char}_{T'}(c_A)$ is the same for all trees T' in a neighborhood of T , and the axis of β passes close to $b_A(T')$ and $b'_A(T')$ in all trees in this neighborhood. This implies that the relative orientations of (α, β) eventually agree in $\rho(T_n)$ and in $\rho(T)$. \square

Proof of Proposition 3.7 in Case 1. For all $T \in \mathcal{U}$ and all $t \in [0, 1]$, we let $H(T, t) := \widehat{H}(f_T, 1 - t)$, with the notation from Proposition 3.1. Since $\rho(T_0) = U \in \mathcal{W}$, Lemma 3.2 implies that $\widehat{H}(f_{T_0}, t) \in \mathcal{U}$ for all $t \in [0, 1]$. In addition, we have $\widehat{H}(f_T, 0) = T$ and $\widehat{H}(f_T, 1) = \rho(T) \in X_{S,U}(T_0)$ for all $T \in \mathcal{V}$, and $\widehat{H}(f_T, t) \in cv_N$ for all $(T, t) \in (\mathcal{V} \cap cv_N) \times [0, 1]$ in view of Remark 1.3. By restricting to a smaller neighborhood \mathcal{V}' of T_0 if necessary, we can then assume that $\widehat{H}(f_T, t) \in \mathcal{U}$ for all $(T, t) \in \mathcal{V}' \times [0, 1]$. \square

We will now explain how to adapt the above construction in the case of an HNN extension. We will present the construction, and leave the proof of Proposition 3.7 to the reader in this case.

Case 2: The free splitting S is the Bass–Serre tree of a decomposition of F_N as an HNN extension $F_N = A*$.

The tree U has one of the two shapes represented on Figure 4, where t denotes a stable letter for the splitting. It comes with two orbits of attaching points, and corresponding elements $c_1, c_2 \in F_N$ (there is a degenerate case where $c_1 = c_2$, represented on the right of Figure 4). If $c_1 = e$, then we let $b'_1(T) := b_1(T)$. If $c_1 \neq e$, then we let $b'_1(T)$ be either the projection of $\text{Char}_T(t^{-1}c_2t)$ to $\text{Char}_T(c_1)$ if these sets have nondegenerate intersection, or else the midpoint of their intersection. Similarly, we let $b'_2(T) := b_2(T)$ if $c_2 = e$, or else $b'_2(T)$ is defined as either the projection of $\text{Char}_T(tc_1t^{-1})$ to $\text{Char}_T(c_2)$ if these sets have nondegenerate intersection, or else as the midpoint of their intersection.

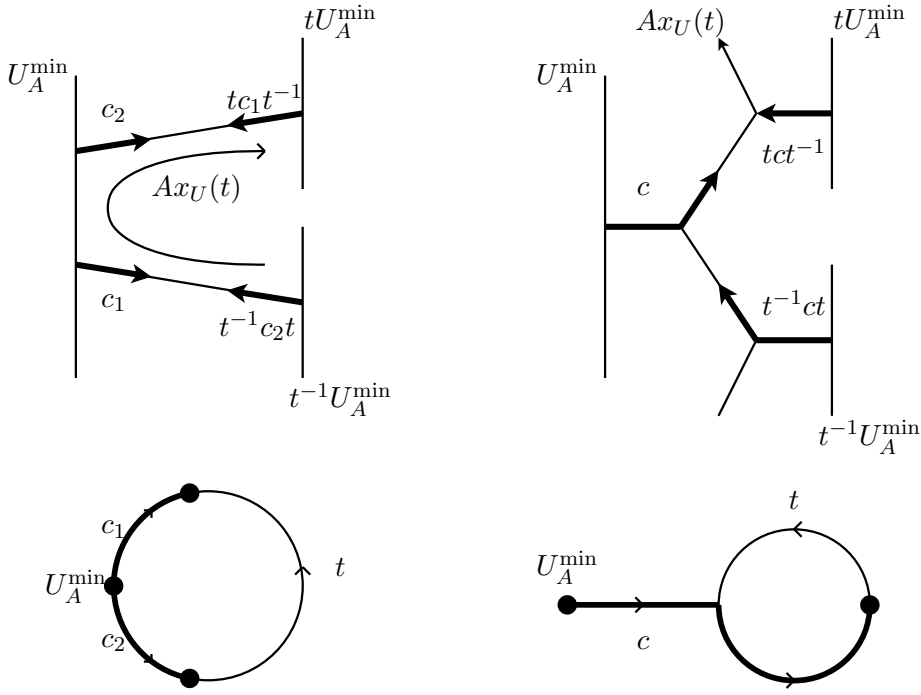


Figure 4: The tree U in Case 2 of the proof of Proposition 3.7, and the quotient graph of groups.

The tree $\rho(T)$ is then defined as the tree that splits as a graph of actions over S , with vertex action T'_A and attaching points $b'_1(T)$ and $b'_2(T)$, where the simplicial edge is given length $d_T(tb'_1(T), b'_2(T))$.

3.5 Approximations by trees in cv_N

Proposition 3.9. *Every point in $\widehat{cv_N}$ is a limit of points in cv_N .*

The proof of Proposition 3.9 is by induction on the rank N of the free group. The same fact with $\widehat{cv_N}$ in place of $\widehat{cv_N}$ was proved in [6, 3], we will follow the argument from [19, Lemma 5.6].

Proof. The case where $N = 1$ is obvious, so we let $N \geq 2$, and we assume by induction that Proposition 3.9 has already been proved for all $N' < N$. In view of Proposition 3.3, it is enough to approximate every tree $U \in \widehat{cv_N}$ that splits as a graph of actions over a one-edge free splitting S of F_N with admissible attaching points, by a sequence of trees in cv_N . For simplicity of exposition, we will assume that S is the Bass-Serre tree of a graph of groups decomposition of F_N as a free product $F_N = A * B$, and leave the case of an HNN extension to the reader. We denote by l the length of the simplicial edge from the graph of actions in U . The vertex trees U_A and U_B may fail to be minimal, and can be obtained in general from their minimal subtrees by adding edges e_A and e_B of length l_A and l_B with nontrivial stabilizers c_A and c_B (with the convention that these edges point outwards the minimal subtrees). Our induction hypothesis enables us to find approximations U_A^n and U_B^n of the minimal subtrees U_A^{\min} and U_B^{\min} of the vertex actions (with approximations of the attaching points that can be assumed to belong to the characteristic sets of c_A and c_B). We then obtain an approximation U_n of U , as a graph of actions with vertex actions U_A^n and U_B^n , where the attaching points are

the unique points on the characteristic sets of c_A (resp. c_B) at distance l_A (resp. l_B) from the basepoint in the positive direction, and the simplicial edge has length l . That $(U_n)_{n \in \mathbb{N}}$ converges to U follows from the argument in [19, Lemma 5.6], see also the proof of Lemma 3.8 of the present paper. \square

3.6 The space $\widehat{CV_N}$ is an AR, and the boundary is a Z -set.

We have already proved that $\widehat{CV_N}$ is a compact, metrizable, finite-dimensional space. The boundary $\widehat{CV_N} \setminus CV_N$ is nowhere dense by Proposition 3.9. It is known in addition that CV_N is contractible [8], and it is also locally contractible. Theorem B.1 from the appendix therefore reduces the proof of the fact that $\widehat{CV_N}$ is an AR, and $\widehat{CV_N} \setminus CV_N$ is a Z -set (Theorem 2 from the introduction), to the proof of the following statement. We recall from the introduction that a nowhere dense closed subset Z of metric space X is *locally complementarily contractible (LCC)* in X if for all $z \in Z$, and every open neighborhood \mathcal{U} of z in X , there exists a smaller neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of z in X such that the inclusion $\mathcal{U} \setminus Z \hookrightarrow \mathcal{V} \setminus Z$ is nullhomotopic.

Theorem 3.10. *The set $\widehat{CV_N} \setminus CV_N$ is LCC in $\widehat{CV_N}$.*

We will actually prove a version of Theorem 3.10 for the closure of the unprojectivized outer space, from which Theorem 3.10 follows.

Theorem 3.11. *The set $\widehat{cv_N} \setminus cv_N$ is LCC in $\widehat{cv_N}$.*

The proof of Theorem 3.11 is by induction on N . The case where $N = 1$ is obvious. From now on, we will let $N \geq 2$, and assume by induction that

(\mathcal{H}_N) Theorem 3.11 has already been established for all $N' < N$.

Let $T_0 \in \widehat{cv_N}$, and let \mathcal{U} be an open neighborhood of T_0 in $\widehat{cv_N}$. Let $X_{S,U}(T_0)$ be the set defined in Section 3.4. Again, for simplicity of exposition, we will assume that S is the Bass–Serre tree of a graph of groups decomposition of F_N as a free product $F_N = A * B$, and leave the case of an HNN extension to the reader. For all T in a sufficiently small neighborhood of $\rho(T_0)$ in $X_{S,U}(T_0)$, there exist trees $T_A^{\min} \in \widehat{cv(A)}$ and $T_B^{\min} \in \widehat{cv(B)}$ (coming with basepoints given by the maps b_A and b_B), and nonnegative real numbers l, l_A, l_B , such that T is obtained in the following way: if c_A is elliptic in T_A^{\min} , then we first define T'_A as the tree obtained from T_A^{\min} by pulling out an edge of minimal length with stabilizer c_A , so that the arc fixed by c_A in T_A has length at least l_A (otherwise we just let $T_A = T_A^{\min}$), and we do the same on the B side. The tree T is the graph of actions over S with vertex actions T_A and T_B , where the simplicial edge from the splitting is assigned length l , and the attaching points are at distance l_A and l_B (in the positive direction) from the basepoints $b_A(T_A^{\min})$ and $b_B(T_B^{\min})$ on the characteristic sets of c_A and c_B .

Remark 3.12. It is crucial here to notice that nonnegativity of l_A follows from the fact that in a neighborhood of $\rho(T_0)$, the relative orientation of $[b_A(T), b'_A(T)]$ and of $\text{Char}_T(c_A)$ is $+1$ as soon as $c_A \neq e$ (the same also holds true on the B side). This fact would fail if we worked in $\widehat{cv_N}$ instead of $\widehat{cv_N}$.

We will then write

$$T = \mathcal{T}(T_A^{\min}, T_B^{\min}, l, l_A, l_B).$$

Notice that the parameters $T_A^{\min}, T_B^{\min}, l, l_A, l_B$ are completely determined by T , and vary continuously with T in a sufficiently small neighborhood of $\rho(T_0)$: we have $l(T) = d_T(b'_A(T), b'_B(T))$, and $l_A(T) := d_T(b_A(T), b'_A(T))$, and $l_B(T) := d_T(b_B(T), b'_B(T))$.

Proposition 3.13. *Assume (\mathcal{H}_N) . Let $T_0 \in \widehat{cv_N}$, and let \mathcal{U} be an open neighborhood of T_0 in $\widehat{cv_N}$. Then $X_{S,U}(T_0) \cap (\widehat{cv_N} \setminus cv_N)$ is LCC at $\rho(T_0)$ in $X_{S,U}(T_0)$.*

Proof. Let \mathcal{U}' be an open neighborhood of $\rho(T_0)$ in $X_{S,U}(T_0)$. One can find neighborhoods \mathcal{U}_A of $(T_0)_A^{\min}$ in $\widehat{cv(A)}$, and \mathcal{U}_B of $(T_0)_B^{\min}$ in $\widehat{cv(B)}$, as well as neighborhoods of $l(T_0), l_A(T_0), l_B(T_0)$ in \mathbb{R} such that all trees built as above with parameters in these neighborhoods, belong to \mathcal{U}' . By our induction hypothesis (\mathcal{H}_N) , there exists an open neighborhood $\mathcal{V}_A \subseteq \mathcal{U}_A$ of $(T_0)_A^{\min}$ in $\widehat{cv(A)}$ such that $\mathcal{V}_A \cap cv(A) \hookrightarrow \mathcal{U}_A \cap cv(A)$ is nullhomotopic, and similarly for B . Let $\mathcal{V} \subseteq \mathcal{U}'$ be such that for all $V \in \mathcal{V} \cap cv_N$, one has $V_A^{\min} \in \mathcal{V}_A$ and $V_B^{\min} \in \mathcal{V}_B$. Then given $V \in \mathcal{V} \cap cv_N$, there is a continuous assignment of paths $V'_A(t)$ from V_A to V'_A (with t varying in $[0, 1]$ and V' varying in $\mathcal{V} \cap cv_N$), staying in $\mathcal{U}_A \cap cv(A)$, and similarly for B . We get a continuous assignment of paths that stay within $\mathcal{U}' \cap cv_N$ from V to any $V' \in \mathcal{V} \cap cv_N$ by letting

$$V'(t) := \mathcal{T}(V'_A(t), V'_B(t), l(t), l_A(t), l_B(t)),$$

where $l(t) := (1-t)l(V) + t.l(V')$, and $l'_A(t) := (1-t)l_A(V) + t.l_A(V')$, and $l_B(t) := (1-t)l_B(V) + t.l_B(V')$. \square

Proof of Theorem 3.11. Let $T_0 \in \widehat{cv_N}$, and let \mathcal{U} be an open neighborhood of T_0 . Using Proposition 3.13, there exists an open neighborhood $\mathcal{W} \subseteq \mathcal{U}$ of $\rho(T_0)$ in $X_{S,U}(T_0)$ such that the inclusion map $\mathcal{W} \cap cv_N \hookrightarrow \mathcal{U} \cap X_{S,U}(T_0) \cap cv_N$ is nullhomotopic. By Proposition 3.7, there exists a neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of T_0 in $\widehat{cv_N}$ such that for all $T \in \mathcal{V} \cap cv_N$, and all $t \in [0, 1]$, one has $\widehat{H}(f_T, t) \in \mathcal{U} \cap cv_N$, and $\rho(T) = \widehat{H}(f_T, 1) \in \mathcal{W} \cap cv_N$. This implies that the inclusion map $\mathcal{V} \cap cv_N \hookrightarrow \mathcal{U} \cap cv_N$ is also nullhomotopic. \square

A A Čech homology lemma

We regard the n -sphere S^n as the boundary of the unit ball B^{n+1} in \mathbb{R}^{n+1} and $S^{n-1} \subset S^n$ as the equator $S^n \cap (\mathbb{R}^n \times \{0\})$. The northern (resp. southern) hemisphere B_+^n (resp. B_-^n) is the set of points in S^n whose last coordinate is nonnegative (resp. nonpositive). We will say that a map $h : S^n \rightarrow \mathbb{R}$ is *standard* if $h^{-1}(0) = S^{n-1}$, $h^{-1}([0, \infty)) = B_+^n$ and $h^{-1}((-\infty, 0]) = B_-^n$. The reader is referred to [11, Chapters IX,X] for basic facts about Čech homology.

Lemma A.1. *Let $n \geq 1$, and let $\tilde{h} : B^{n+1} \rightarrow \mathbb{R}$ be a continuous map whose restriction to $S^n = \partial B^{n+1}$ is standard. Then the inclusion*

$$S^{n-1} \hookrightarrow \tilde{h}^{-1}(0)$$

is trivial in Čech homology \check{H}_{n-1} with $\mathbb{Z}/2$ -coefficients (one has to consider reduced homology if $n = 1$).

Proof. The proof is illustrated in Figure 5. Take a triangulation of B^{n+1} such that S^n and S^{n-1} are subcomplexes and construct from it a handle decomposition, see e.g. [28, Proposition 6.9]. Let M be the union of the handles that intersect $Y := \tilde{h}^{-1}(0)$. Then M is a compact PL manifold with boundary, it contains $Y \setminus S^{n-1}$ in its interior, and

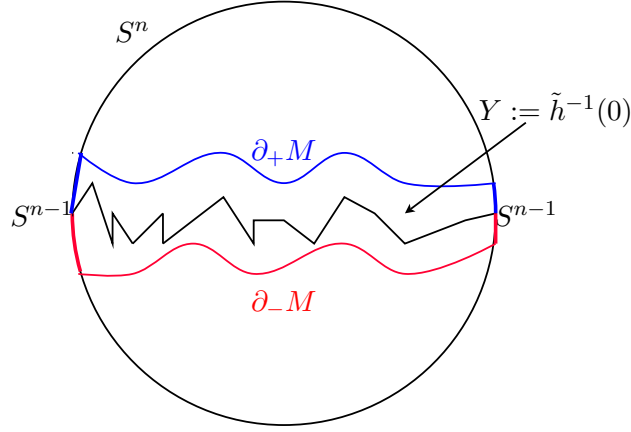


Figure 5: The situation in the proof of Lemma A.1 (illustrated on the figure in the case where $n = 1$). The manifold M is the region inbetween the blue and red curves, which correspond to $\partial_+ M$ and $\partial_- M$, respectively.

∂M can be written as the union $\partial_+ M \cup \partial_- M$ of two codimension 0 submanifolds that intersect in S^{n-1} , the boundary of each. Namely, put

$$\partial_+ M = \partial M \cap \tilde{h}^{-1}([0, \infty))$$

and

$$\partial_- M = \partial M \cap \tilde{h}^{-1}((-\infty, 0]).$$

That these are manifolds with boundary follows from the assumption that $\tilde{h}|_{S^n}$ is standard and the fact that by construction $\partial M \cap S^n$ contains a neighborhood of S^{n-1} in S^n . In particular, the inclusion

$$S^{n-1} \hookrightarrow M$$

is trivial in (singular) H_{n-1} , since S^{n-1} bounds a compact n -manifold $\partial_+ M$.

If the triangulation of B^{n+1} we start with has sufficiently small simplices, then M will be contained in a prechosen neighborhood of Y . Therefore we can represent Y as the nested intersection of compact manifolds M_i , and since by the continuity of Čech homology [11, Theorem X.3.1], we have

$$\check{H}_{n-1}(Y) = \varprojlim H_{n-1}(M_i),$$

the assertion follows since $H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(M_i)$ is trivial for each i . \square

B Proving that a space is an AR: a topological criterion

In this appendix, we establish the ANR criterion that suits our purpose. Recall from the introduction that a metric space X is *locally n -connected* (LC^n) if for every $x \in X$ and every open neighborhood \mathcal{U} of x , there exists an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that the inclusion $\mathcal{V} \hookrightarrow \mathcal{U}$ is trivial in π_i for all $0 \leq i \leq n$. A nowhere dense closed subset $Z \subseteq X$ is *locally complementarily n -connected* (LCC^n) in X if for every $z \in Z$ and every open neighborhood \mathcal{U} of z in X , there exists a smaller neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of z in X such that the inclusion $\mathcal{V} \setminus Z \hookrightarrow \mathcal{U} \setminus Z$ is trivial in π_i for all $0 \leq i \leq n$.

It is a classical fact that a compact n -dimensional LC^n metrizable space is an ANR [20, Theorem V.7.1], and we need the following extension. The methods we use are classical, but we could not find this statement in the literature, so we include a proof. We use [20] as a reference, but the techniques were established earlier, see [22, 23, 5, 10].

Theorem B.1. *Let X be a compact metrizable space with $\dim X \leq n$, and let $Z \subset X$ be a nowhere dense closed subset which is LCC^n in X and such that $X \setminus Z$ is LC^n .*

Then X is an ANR and Z is a Z -set in X .

If in addition $X \setminus Z$ is assumed to be contractible, then X is an AR.

Proof. The key is to prove the following statement.

Claim. Let Y be a compact metrizable space, with $\dim Y \leq n + 1$, let $A \subset Y$ be a closed subset, and let $f : A \rightarrow X$ be a map. Then there exists an open neighborhood \mathcal{O} of A in Y and an extension $\tilde{f} : \mathcal{O} \rightarrow X$ of f such that $\tilde{f}(\mathcal{O} \setminus A) \subseteq X \setminus Z$.

We start by explaining how to derive Theorem B.1 from the claim. To prove that Z is a Z -set, take $Y = X \times [0, 1]$ and $A = X \times \{0\}$, and let $f : A \rightarrow X$ be the identity. Then an extension to a neighborhood produces an instantaneous homotopy of X off of Z , so Z is a Z -set (indeed, since X is compact, a neighborhood of $X \times \{0\}$ includes $X \times [0, \varepsilon]$ for some $\varepsilon > 0$, and we can reparametrize to $X \times [0, 1]$).

We now argue that X is LC^n , and this will establish that X is an ANR by [20, Theorem V.7.1]. We only need to show that X is LC^n at every point $z \in Z$. Let $z \in Z$ and let \mathcal{U} be a neighborhood of z in X . Choose a neighborhood \mathcal{V} of z in X such that $\mathcal{V} \setminus Z \hookrightarrow \mathcal{U} \setminus Z$ is trivial in π_i for all $i \leq n$. Let $f : S^i \rightarrow \mathcal{V}$ be a given map. Using the instantaneous deformation, we can homotope f to $f' : S^i \rightarrow \mathcal{V} \setminus Z$ within \mathcal{V} . But now f' is nullhomotopic within \mathcal{U} by assumption.

It remains to prove the claim. Choose an open cover \mathcal{W} of $Y \setminus A$ whose multiplicity is at most $n + 2$ and so that the size of the open sets gets small close to A . For example, one could arrange that if $y \in W \in \mathcal{W}$ then $\text{diam} W < \frac{1}{2}d(y, A)$ with respect to a fixed metric d on Y (such coverings are called *canonical*, see e.g. [20, II.11]). Next, let N be the nerve of \mathcal{W} , thus $\dim N \leq n + 1$. There is a natural topology on $A \cup N$ where A is closed, N is open, and a neighborhood of $a \in A$ induced by an open set $\mathcal{O} \subset Y$ with $a \in \mathcal{O}$ is $\mathcal{O} \cap A$ together with the interior of the subcomplex of N spanned by those $W \in \mathcal{W}$ contained in \mathcal{O} . For more details see e.g. [20, II.12].

Since there is a natural map $\pi : Y \rightarrow A \cup N$ that takes A to A and $Y \setminus A$ to N (given by a partition of unity) it suffices to prove the claim after replacing Y by $A \cup N$.

The extension is constructed by induction on the skeleta of N , and uses the method of e.g. [20, Theorem V.2.1]. The only difference is that we want in addition $\tilde{f}(\mathcal{O} \setminus A) \subseteq X \setminus Z$.

To extend $f : A \rightarrow X$ to the vertices of N , use the assumption that Z is nowhere dense in X to send a vertex v close to some $a \in A$ to a point in $X \setminus Z$ close to $f(a)$. Inductively, suppose $0 \leq i \leq n$, and f has been extended to the i -skeleton of some subcomplex N_i of N , in such a way that

- $f(N_i) \subseteq X \setminus Z$,
- $A \cup N_i$ contains a neighborhood of A in $A \cup N$, and
- for every $\delta > 0$, there exists a neighborhood $\mathcal{U}_{i,\delta}$ of A in $A \cup N_i$ such that the f -image of every i -simplex of N_i contained in $\mathcal{U}_{i,\delta}$ has diameter at most δ .

Since X is compact and Z is LCC^n in X , there exists $\delta_{i+1} > 0$, and a function $r : (0, \delta_{i+1}) \rightarrow \mathbb{R}_+$, with $r(t) \rightarrow 0$ as t decreases to 0, such that any map $\phi : S^i \rightarrow X \setminus Z$ whose image has diameter at most $d < \delta_{i+1}$, extends to a map $\tilde{\phi} : B^{i+1} \rightarrow X \setminus Z$ whose image has diameter at most $r(d)$. Now apply this to every $(i+1)$ -simplex in N_i such that the image of its boundary has diameter strictly smaller than δ_{i+1} . When extending to the simplex, always arrange that the diameter of the image is controlled by the function r . This completes the inductive step. \square

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